§5.3 Bézout's theorem
贝祖

Thm: F, G = proj. plane curves. $m = \deg F$, $n = \deg G$.

$$\gcd(F,G)=1 \Rightarrow \sum_P I(P, F\cap G) = mn$$

Pf: $\#(F\cap G) < \infty$ WMA $(F\cap G)\cap H_\infty = \phi$ (by coordinate change)

$$\Rightarrow \sum_P I(P, F\cap G) = \sum_P I(P, F_* \cap G_*) = \dim_k k[X,Y]/(F_*, G_*).$$

$\Gamma_* = k[X,Y]/(F_*, G_*)$, $\Gamma = k[X,Y,Z]/(F,G)$ $R = k[X,Y,Z]$

$\Gamma_d = \{\text{forms of deg } d\} \subseteq \Gamma$

$R_d = \{\text{forms of deg } d\} \subseteq R$

ONTS: $\dim \Gamma_* = \dim \Gamma_d$ & $\dim \Gamma_d = mn$ for $d \gg 0$.

Step 1: $\dim \Gamma_d = mn$ for all $d \geq m+n$.

$$0 \to R \xrightarrow{\psi} R\times R \xrightarrow{\varphi} R \xrightarrow{\pi} \Gamma \to 0$$

$C \mapsto (GC, -FC)$ $\quad (A,B) \mapsto AF+BG$

$\gcd(F,G)=1 \Rightarrow$ exact

$$\Rightarrow 0 \to R_{d-m-n} \xrightarrow{\psi} R_{d-m} \times R_{d-n} \xrightarrow{\varphi} R_d \xrightarrow{\pi} \Gamma_d \to 0 \text{ exact}$$

$$\dim R_d = \frac{(d+1)(d+2)}{2} \Rightarrow \dim \Gamma_d = mn \quad (\text{if } d \geq m+n)$$

Step 2: $\alpha : \Gamma \hookrightarrow \Gamma \quad \overline{H} \mapsto \overline{ZH}$

NTS: $ZH = AF + BG \Rightarrow H = A'F + B'G$. for some A', B'.

$\forall\ J \in k[X,Y,Z] \quad J_0 := J(X,Y,0)$

$$\left.\begin{array}{l} F \cap G \cap Z = \phi \Rightarrow \gcd(F_0, G_0) = 1 \\ Z \mid AF + BG \Rightarrow A_0 F_0 = -B_0 G_0 \end{array}\right\}$$

$$\Rightarrow \begin{cases} B_0 = F_0 C \\ A_0 = -G_0 C \end{cases} \quad \text{for some } C \in k[X,Y]$$

$$\begin{array}{l} A_1 := A + CG \\ B_1 := B - CF \end{array} \quad \left(\Rightarrow \begin{array}{l} (A_1)_0 = A_0 + CG_0 = 0 \\ (B_1)_0 = B_0 - CF_0 = 0 \end{array} \right)$$

$$\Rightarrow \begin{cases} A_1 = ZA' \\ B_1 = ZB' \end{cases}$$

$$\Rightarrow ZH = AF + BG = A_1F + B_1G$$

$$= Z(A'F + B'G)$$

$$\Rightarrow H = A'F + B'G.$$

Step 3. $d \geq mn$, $A_1, \cdots, A_{mn} \in R_d$ a lifting of a basis of Γ_d. $A_{i*} := A_i(X,Y,1) \in k[X,Y]$ $(R_d \twoheadrightarrow \Gamma_d)$

$a_i := A_{i*} \pmod{*} \in \Gamma_*$

Then $a_1, \cdots, a_{mn}$ forms a basis for Γ_*.

Step 2 $\Rightarrow \Gamma_d \xrightarrow[\cong]{\alpha} \Gamma_{d+1}$ for $d \geq mn$.

$\Rightarrow Z^r A_1, \cdots, Z^r A_{mn}$ basis for Γ_{d+r} $\forall r \geq 0$.

a_i generate Γ_*: $\forall\, h = \overline{H} \in \Gamma_*$ $H \in k[X,Y]$

$\Rightarrow Z^N H^* =$ form of degree $d+r$ (for some N)

$$\Rightarrow Z^N H^* = \sum_{i=1}^{mn} \lambda_i Z^r A_i + BF + CG$$

$$\overset{(\cdot)_*}{\Rightarrow} H = \sum_{i=1}^{mn} \lambda_i (A_i)_* + B_* F_* + C_* G_*$$

$$\Rightarrow h = \sum_{i=1}^{mn} \lambda_i a_i \Rightarrow \checkmark$$

a_i are independent: if $\sum \lambda_i a_i = 0$

$$\Rightarrow \sum \lambda_i (A_i)_* = BF_* + CG_*$$

$$\Rightarrow Z^r \sum \lambda_i A_i = Z^s B^* F + Z^t C^* G$$

$$\Rightarrow \sum \lambda_i \overline{Z^r A_i} = 0 \in \Gamma_{d+r}$$

$$\Rightarrow \lambda_i = 0 \ \forall i \quad \left(\{\overline{Z^r A_i}\} = \text{basis} \right)$$

Cor. 1) $\gcd(F,G)=1 \Rightarrow \sum_P m_P(F)\, m_P(G) \leq \deg F \cdot \deg G$.

2) $\gcd(F,G)=1$ and $\# F \cap G = \deg F \cdot \deg G$

$\Rightarrow \forall P \in F \cap G$ is simple on F & G.

3). $\# F \cap G > \deg F \cdot \deg G \Rightarrow \gcd(F,G) \neq 1$.

§5.4 Multiple Points

F = irr. curve of deg. n.
$m_P :=$ multiplicity of F at P.

Prob 5.22 $\Rightarrow \sum \frac{m_P(m_P-1)}{2} \leq \frac{n(n-1)}{2}$

prob 5.8, 5.13 $\Rightarrow$ not optimal

Thm 2. F = irr of deg n. then

$$\sum \frac{m_P(m_P-1)}{2} \leq \frac{(n-1)(n-2)}{2}$$

Pf: $r := \frac{(n-1)(n-1+3)}{2} - \sum \frac{(m_P-1)\, m_P}{2} \geq 0$.

choose $Q_1, \cdots, Q_r$ simple pts on F.

§5.2 Thm 1 $\Rightarrow \exists$ G of deg n-1 s.t. $\begin{cases} m_P(G) \geq m_P - 1, \ \forall P. \\ m_{Q_i}(G) \geq 1. \end{cases}$

Bézout thm $\Rightarrow n(n-1) \geq \sum m_P \cdot (m_P - 1) + r \Rightarrow$ ✓

Cor: 1) lines & conics are nonsingular
2) irr. cubic has at most one double pt.
3) irr. quartic has at most three double pts or one triple pt.

Example (thm is optimal.) $F = X^n + Y^{n-1}Z \Rightarrow m_{[0:0:1]} = n-1$.